Diagrams in Mathematical Education: A Philosophical Appraisal

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Computer-assisted learning environments in education are not just a futuristic dream but a reality, and the trend seems to be unstoppable. Mathematics is no exception, prompting psychologists in the field of mathematics education to investigate computers in pedagogy and learning, especially visual components of mathematical learning.[1] This essay addresses an epistemological issue that is posed in using diagrams in mathematical reasoning: What is the epistemological status of diagrammatic reasoning? Specifically, is diagrammatic reasoning a full-fledged form of mathematical reasoning or is it merely an intuitive appendage (a pedagogical or psychological helping hand for the student rather than true mathematical reasoning that, it is typically thought, is characterized by purely abstract or logical thought)? This issue especially pertains to the work of education theorists who hold that diagrams are purely intuitive aids, yet seem to implicitly accept them as legitimate components of proofs, for example, of the Pythagorean theorem.[2]

It is important to distinguish the issue of the epistemological status of mathematic reasoning with diagrams from a developmental issue. Typically, in theories of mathematical development a series of stages is theorized that to one degree or another a student is thought to pass through. Even though theorists may argue, for example, over whether one particular stage is a prerequisite for another and, if so, to what extent, there is no disagreement that stages are required (for example, there are the stages of Piaget and those of Van Heil[3]). This paper does not dispute this contention. Rather it questions whether diagrammatic mathematical reasoning can legitimately be judged to be *merely* intuitive, even though it may be a prerequisite for other forms of mathematical reasoning.

This essay does not take the view that mathematical diagrammatic reasoning can be divorced from linguistic communication about diagrams or from linguistically expressed inferences about the diagrams. So there is no argument here for the position that visual cognition, if there is such a thing separate from other forms of cognition, alone is sufficient for mathematical reasoning.

With these clarifications in hand, this essay argues that diagrammatic reasoning is not just an intuitive tool, but a full-fledged part of some types of mathematical reasoning. It does so by examining the informational and representational function of diagrams in mathematical proofs.

Diagrams and Mathematical Thought

To focus the discussion, a diagrammatic proof of the Pythagorean theorem is considered. In fact, a special case of the theorem is first discussed in which the theorem is proved for a right-angle, isosceles triangle. This is the proof that Plato made famous in *Meno* where a boy, under Socrates' instruction, has a series of insights that allow him to figure out how to double a square (how to construct a square that is exactly double the area of any given square).[4] By Socrates' lights, the boy's performance demonstrates Socrates' theory of learning as "remembering," a sort of innateness claim: "[T]here is no teaching but recollection" (82a). For the purposes of this essay, the interest of the scene lies in its true-to-life dramatization of linguistic and mathematical prerequisites for a certain type of reasoning and crucial features of that reasoning.

The linguistic prerequisites for the reasoning are implicitly given up front:

S: Is he a Greek? Does he speak Greek?
M: Very much so. He was born in my household (82b).

So the boy is proficient in Greek and, of course, considerable language understanding is necessary to follow Socrates' argument. The boy also knows some mathematics. He knows, for example, what a square is: "S: Tell me now, boy, you know that a square figure is like this? — I do (82b)." and he knows four is double two (85b). In fact, Socrates and the boy share many concepts; otherwise they could not converse with each other.

Relying on the particular figures in the sand, the boy learns, through a series of sound inferences, aided along the way by figuring out the error with a few unsound inferences, how to double a particular square in the sand and, generalizing from this knowledge, how to double any square. This generalization, it seems, relies on both Socrates and the boy taking the properties of the diagrams in the sand to represent the relevant properties of all squares. The text suggests this by Socrates' query: "And such a figure [the square drawn in the sand] could be larger or smaller?" and the boy's response: "Certainly." (*FD*, 82c) So the reasoning of the boy, once he finds the solution for the particular square drawn in the sand, if spelled out, would presumably be something like this inference, labeled INF for future reference:

(INF) *This particular square is like any other except for size, location, and orientation; the technique for doubling this square does not depend on its size, location, or orientation; therefore, this technique for doubling this square will work for any square.*

The contention that Plato can be interpreted as characterizing the boy's thought in this way is, admittedly, not a knock-down exegetical argument, nor is it intended to be. Plato is being drawn on for his true-to-life account of a mathematical learning experience. Even if the appeal to INF by reasoners does not follow from an exegetical account of Plato, still, what matters is that reasoners employ INF in the problem dramatized by Plato. This is commonly accepted; in fact, inferences such as INF are a basis of geometric reasoning.

Since, in effect, the technique for doubling a square, learned by the servant boy, is a proof of a special case of the Pythagorean theorem for isosceles, right-angle triangles, this scene shows how diagrams can play a role in mathematical proofs. Diagrams can play a similar role in proving the full Pythagorean theorem.[5]

Such proofs, like all diagrammatic reasoning, are held in disrepute by most mathematicians and logicians. At best they are considered mere "aids to intuition" and not legitimate constituents of mathematical proofs, due to the danger of generalizing from accidental features of diagrams. Logician Neil Tennant adheres to this standard view:

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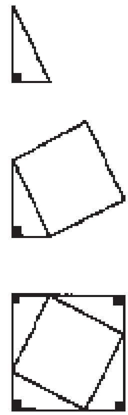
> [The diagram] is only an heuristic to prompt certain trains of inference.…it is dispensable as a proof-theoretic device; indeed…it has no proper place in the proof as such. For the proof is a syntactic object consisting only of sentences arranged in a finite and inspectable array.[6]



Logicians Jon Barwise and John Etchemendy contend that this stance is mistaken, that if care is taken not to use accidental features of diagrams in proofs, reasoning with diagrams can be sound. In challenging the current dogma that all sound reasoning must be exclusively sentential, they draw on the work of psychologists such as Stenning and Kosslyn, who similarly have challenged this dogma.[7] Barwise and Etchemendy recommend a theory of "heterogeneous" reasoning that specifies that thought is not characterized by any one representationalist system. To illustrate their point they present an elegant proof of the Pythagorean theorem that employs both formal algebraic manipulation and diagrammatic manipulation.[8] This is a standard proof, the same as the following, see figure at right.

The task is to show, starting with an arbitrary right-angle triangle with sides a, b, and c, that $a^2 + b^2 = c^2$. First construct a square on the hypotenuse c and replicate the triangle three time as shown. Since the sum of the angles of a triangle is a straight line, it can be easily shown that *ABCD* is a square. Now the area of *ABCD* can be computed in two ways. Since the sides of *ABCD* are all $(a + b)$ in length, its area is $(a + b)^2 = a^2 + 2ab + b^2$. Alternatively, the area is the sum of the areas of the four triangles plus the area of the central square. That is, the area is $2ab + c^2$. These two calculations yield $a^2 + b^2 = c^2$, the desired result.

Having presented this proof, Barwise and Etchemendy argue:

> It seems clear that this is a legitimate proof of the Pythagorean theorem. Note, however, that the diagrams play a crucial role in the proof. We are not saying that one could not give an analogous (and longer) proof without them, but rather that the proof as given makes crucial use of them. To see this, we only need note that without them, the proof given above makes no sense.
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> This proof of the Pythagorean theorem is an interesting combination of both geometric manipulation of a diagram and algebraic manipulation of nondiagrammatic symbols. Once you remember the diagram, however, the algebraic half of the proof is almost transparent. This is a general feature of many geometric proofs: Once you have been given the relevant diagram, the rest of the proof is not difficult to figure out. It seems odd to forswear nonlinguistic representation and so be forced to mutilate this elegant proof by constructing an analogous linguistic proof, one no one would ever discover or remember without the use of diagrams.[9]

The Dual Function of the Diagrams

The nature of mathematical reasoning that relies on mathematical diagrams can be ascertained by asking: What is the function of diagrams in mathematical proofs (that use them)? The diagram in the scene from the *Meno* seems to indicate two interrelated types of functions. First, the diagram performs a function as a particular diagram that Socrates and the boy talk about and point at, that Socrates draws, and that the boy reasons about. In the end the boy discovers a property of this particular diagram of a square.

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Second, the diagram performs a function as a representation that allows for the generalization (such as that involved with INF) from the particular inferences concerning the diagram in the sand, as noted above. It would seem that (with the caveat regarding INF mentioned earlier) that agreement that the figure drawn in the sand is a square and "such a figure could be larger or smaller" allows Socrates, at the end of the lesson, to claim that the boy learned (that is, "remembered") how to double any square for the boy learned on what line any square can be doubled (82c): "That is, on the line that stretches from corner to corner of the four-foot figure? — Yes. — Clever men call this the diagonal, so that if diagonal is its name, you say that the double figure would be that based on the diagonal? — Most certainly, Socrates (85b)." A particular square not only represents any square but represents in a special way that allows for the inference INF to any square. The particular square is the same (approximately) as all squares except for size, location, and orientation. Thus, if INF is to apply, the choice of possible representations of squares is narrowed down to a single type: a particular square of arbitrary size, location, and orientation. For no other type of representation is the same as all squares except for size, location, and orientation. (A further reasonable restriction on the representation of squares is that it must be visually perceived in one viewing.) Thus, although we can assign almost anything to represent squares, in this situation only one type of thing will do, an actual square in medium. Even though such a square need not be all that accurate, it needs to be accurate enough to support the type of diagrammatic reasoning displayed by the boy.

Diagrams function in this dual way in other types of reasoning. Suppose I set up a chess problem *P* on my chess board. Say the problem is of the sort that is typically found in chess columns in newspapers where a position is given, and the reader is asked to find a checkmate in two moves. If I solve the particular chess problem that I have set up on my chess board, why can I correctly infer that I have found a solution to the problem that other people worked on when they set up *P* on their own chess boards (assuming that they and I have set up the problem correctly)? The correctness of this inference is due to the configuration of chess pieces on my chess board that effectively *represents* the problem *P* by being the same configuration of pieces of an 8 x 8 grid of a chess board as *P* and hence the same problem that others are working on. Since my solution depends on this configuration on a 8 x 8 grid only and not, for example, on the size of my chess board or the design of my chess pieces, my solution will work for all.

The chess problem example also highlights the role of manipulation of particular diagrams. Merely setting up the problem is not enough to solve it. One moves pieces here and there or imagines moving them, and checks to see if the new configuration leads to mate in one more move. If successful, a property of the configuration of the chess pieces on the chess board — that the position does lead to mate in two — is inferred from known properties of the configuration.

Similarly, the boy, aided by Socrates, checks various squares to ascertain whether they are twice the size of the original square upon which they were constructed. In the end, the boy discovers a property of the square in the sand

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unknown to him beforehand: A square with a side equal to the original square's diagonal is twice the area as the original square. Once again, an unknown property for the boy is inferred from perceived properties.

Barwise and Etchemendy make the same point in a somewhat different way. For them "[i]nference…is the task of extracting information implicit in some explicitly presented information."[10] Here, presumably, Barwise and Etchemendy are using "information" to refer to conceptual properties of the diagrams rather than the mere intake of sensory neurons.

Thus mathematical diagrams play two interrelated roles in mathematical thought, 1) as objects that people attempt to extract implicit information from, and 2) as representations of classes of geometric shapes to which reasoning applies. Both these elements are indispensable in the reasoning itself. The implicit information is not given in any other form, and the representational role of diagrams is necessary for the proof.

An objection may be that alternate proofs dispense with diagrams. However, that does not undermine the full-fledged role of diagrams in proofs that utilize them, as Barwise and Etchemendy point out. In these situations they can be, and often are, much more than mere aids to intuition.

Implications for Mathematical Education

This position, although controversial in the philosophy of mathematics, is deceptively difficult to avoid in mathematical education theory. For what are teachers doing when they present a diagrammatic proof of the Pythagorean theory other than presenting a full-fledged, honest-to-goodness proof, not a mere aid to intuition of a proof? In fact, teachers claim to their students that it is a proof.

In the psychological literature on mathematics education there is ambiguity concerning the epistemologist status of reasoning with diagrams. For example, Raymond Duval writes: "The usefulness of geometrical figures in the resolution of a problem of geometry is beyond doubt. They provide an intuitive presentation of all constituent relations of a geometrical situation."[11] It is not clear from this quotation what status Duval gives diagrams in proofs, whether the role of mere intuitive aids or a fuller role, because his phrase "provide an intuitive presentation" is ambiguous.[12] In the same article, Duval offers a standard diagrammatic proof of the Pythagorean theorem, much the same as that given by Barwise and Etchemendy above. So one might suppose that, despite ambiguity, he advocates a fuller role for diagrams than that of mere aids to intuition, although this is not clear from what he has written.

Whether or not the question of the epistemological status of mathematical diagrams impinges on research in mathematical education is hard to tell. Nonetheless, the question does pertain to the nature of the activity of educators. Are teachers who employ diagrams in proofs teaching real proofs or just aids to intuition? Although many teachers probably somehow think they are doing only the latter, in reality they are often practicing the former. Similarly, students who learn diagrammatic mathematical proofs are not merely employing intuition; they are doing real mathematics.

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1. See, for example, Rina Hershkowitz, "Psychological Aspects of Learning Geometry" in *Mathematics and Cognition*, ed. Pearla Nesher and Jeremy Kilpatrick (Cambridge, U.K.: Cambridge University Press, 1990), 70-95.

2. Except for discussing the role of diagrams in proofs, this paper does not address the long discussion in mathematics and philosophy of mathematics about the nature of a mathematical proof. Rather, it addresses and comments on the claims of Barwise and Etchemendy in "Visual Information and Valid Reasoning" in *Visualization in Teaching and Learning Mathematics*, ed. W. Zimmerman and S. Cunningham (Washington, DC: Mathematical Association of America, 1991), 9-24. They have raised the issue of the role of diagrams in proofs within the field of mathematics education. Nonetheless, this paper does tacitly accept a common view of the nature of proofs. This view is eloquently expressed by the mathematician G. H. Hardy in "Mathematical Proof," *Mind* (1929): XXXVIII (149), 18: "I have myself always thought of a mathematician as in the first instance an observer, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak he believes that it is there simply because he sees it. It he wishes someone else to see it, *he points to it*, either directly or through the chain of summits which lead him to recognize it himself. When his pupil also sees it, the research, the argument, the *proof* is finished.…The analogy is a rough one, but I am sure that it is not altogether misleading. If we were to push it to its extreme we should be led to a rather paradoxical conclusion; that there is, strictly, no such thing as mathematical proof; that we can, in the last analysis, do nothing but *point*; that proofs are what Littlewood and I call *gas*, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils. This is plainly not the whole truth, but there is a good deal in it. The image gives us a genuine approximation to the processes of mathematical pedagogy on the one hand and of mathematical discovery on the other; it is only the very unsophisticated outsider who imagines that mathematicians make discoveries by turning the handle of some miraculous machine."

3. J. Piaget and B. Inhelder, *The Child's Conception of Space* (London: Norton, 1967) and H. Freudenthal, *Mathematics as an Educational Task* (Dordrecht, The Netherlands: Reidel, 1983).

4. *Plato: Five Dialogues* , trans. G.M.A. Grube (Indianapolis: Hackett, 1981), 81e - 86b.

5. See J. Barwise and J. Etchemendy, "Visual Information."

6. Neil Tennent, "The Withering Away of Formal Semantics," *Mind and Language* (1986): 1, 302-18, quoted by Barwise and Etchemendy, "Visual Information," 9.

7. Keith Stenning, "On Remembering How to Get There: How We Might Want Something Like a Map" in *Cognitive Psychology and Instruction,* eds. A. M. Lesgold, J. W. Pellegrino, S. W. Fokkema, and R. Glaser (New York: Plenum Press, 1977) and Stephen M. Kosslyn, *Image and Mind* (Boston: Harvard University Press, 1980). Here is what Jon Barwise says about this paper on his internet web page <www.phil.indiana.edu/~barwise/barwise,html "It is always fun to work with John Etchemendy. This paper argues for the rehabilitation of diagrams as legitimate tools in proofs. I think it will turn out to be one of the most important things I have contributed to, though most of my logician friends think I'm wrong. We'll see."

8. Barwise and Etchemendy, "Visual Information," 12.

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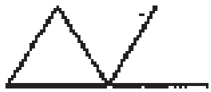
9. Ibid., 12. A similar view toward diagrammatic reasoning in mathematical thought is held by Brown, *Philosophy of Mathematics: Introducing the World of Proofs and Pictures* (London: Routledge, 1998). See also Michael Friedman, *Kant and the Exact Sciences* (Cambridge: Harvard University Press, 1992), where he gives an account of the position of Kant which indicates that Barwise and Etchemendy are somewhat in accord with Kant. Friedman reconstructs a proof (p. 51), given by Kant, that the sum of the angles of a triangle is 180° using the following diagram:



Kant [gives] the standard Euclidean proof of the proposition that the sum of the angles of a triangle=180° = two right angles.…Given a triangle *ABC*, one prolongs the side *BC* to *D* and then draws *CE* parallel to *AB*.…One then notes that r = r_ and r = r_, so r + r + = r_ + = 180°_. Q.E.D.

Friedman then notes: "In contending that construction in pure intuition is essential to this proof, Kant is making two claims that strike us as quite outlandish today. First, he is claiming that (an idealized version of) the figure we have drawn is necessary to the proof. The lines *AB*, *BD*, *CE*, and so on are indispensable constituents; without them the proof simply could not proceed. So geometric proofs are themselves spatial objects. Second, it is equally important to Kant that the lines in question are actually drawn or continuously generated, as it were. Proofs are not only spatial objects, they are spatio-temporal objects as well" (51-52).

10. Jon Barwise and John Etchemendy, "Heterogeneous Logic," in *Logical Reasoning with Diagrams*, ed. G. Allwain and Jon Barwise (New York: Oxford University Press, 1996), 180.

11. Raymond Duval, "Geometric Pictures: Kinds of Representation and Specific Processing," in *Exploiting Mental Imagery with Computers in Education*, ed. Rosamund Sutherland and John Mason (Berlin, Ger.: Springer, 1995), 143.

12. Ibid., 152

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